SOME TOPICS OF SCHOOL MATHEMATICS OF SPECIAL IMPORTANCE TO THOSE STUDENTS WHO EXPECT TO STUDY ANALYTIC GEOMETRY AND CALCULUS.*

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The writer has, for some time, been teaching analytic geometry and calculus to engineering freshmen; and it is his aim, in this paper, to call attention to certain subjects in which his students have seemed to be pretty generally deficient. The topics have been selected somewhat at random, and probably others matters of equal importance have been passed over.

In geometry the essential thing is the development of the student's ability to reason accurately and logically, and this presents one of the big problems of school mathematics—a problem too big for discussion in such a paper as this. In addition to this mental training, the student should acquire a thorough familiarity with the rules for areas, volumes, etc., and with the rules of proportionality of similar figures.

The first essential in algebra is a thorough drill in manipulation. The student should, by much practice, acquire *facility* in factoring, in the insertion (as well as the removal) of parentheses, and in handling simple and complex fractions and radicals. He should not be so easily discouraged as he is when, for example, he has occasion to determine whether or not the equation

$$(5-\sqrt{2})x+(4\sqrt{2}-3)y+1-7\sqrt{2}=0$$

is satisfied by the values

$$x=-\frac{5\sqrt{2}}{2} \quad \text{and} \quad y=\frac{12+3\sqrt{2}}{2}$$

In such work as this, students should learn to save time, without any loss of accuracy, by doing more of the details mentally

* Abstract of a paper read before the Syracuse Section.

and writing less. For example, to solve the equation

$$2x^2-3xy+4y^2-2x+7y-11=0,$$

for x in terms of y, only three steps are necessary. Nothing need be written but

$$2x^2-(3y+2)x+(4y^2+7y-11)=0,$$

$$x=\frac{3y+2\pm\sqrt{9y^2+12y+4-32y^2-56y+88}}{4}$$

$$=\frac{3y+2\pm\sqrt{-23y^2-44y+92}}{4}$$

This example suggests several other things. The student should understand certain algebraic language—he should know what is meant when he is asked to "solve an equation for x in terms of y" or to "arrange an equation as a quadratic in x." Here also is one of the cases where we need the process of inserting parentheses.

As a regular working method for quadratic equations, the formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

should be used. The method of completing the square should be taught, of course, and the formula derived by means of it; but it is wearisome to use this method when one has occasion to solve many quadratic equations, as in analytic geometry. It is also important that the relation between the discriminant b^2-4ac and the nature of the roots of the equation should be more thoroughly mastered.

It is surprising that so many students learn formal methods of solving equations without realizing what it is that they are finding. Many fail to grasp the simple idea that a root of an equation in x is a number which, substituted for x, satisfies the equation; or that a solution of a set of simultaneous equations in several unknown quantities consists of a *set* of values which, substituted respectively for the unknown quantities, will satisfy all the equations. In the ordinary processes of solution there is always danger of obtaining values of the unknown quantities

which are not roots; and hence the only safe plan is to test all results by substitution in the original equations. Thus, by the ordinary processes, one is likely to get

$$x = 1 \text{ or } -4/7$$

from the equation

$$\frac{6x}{x^2-1} - \frac{3}{x-1} = 7,$$

but $x = 1$ is *not* a root, because it does not satisfy the equation.

This example leads to another matter. The idea should be strongly emphasized that *there is no such thing as division by zero,* that such expressions as $5/0$ and $0/0$ are meaningless, and that such expressions as $\frac{3}{x-1}$ and $\frac{x^2-1}{x-1}$ are meaningless when $x = 1$. There is no such number as ∞, and it is a pity that the symbol occurs in our books. To say that $5/0 = \infty$ has no meaning from any point of view whatever. And when we say that $\frac{3}{x-1} = \infty$ when $x = 1$, we are using figurative language which is unfortunate but which we have to make the best of while it remains in our text-books. What we mean is really quite simple. We are not considering the case of $x = 1$, but are letting x *approach* 1, and the fraction $\frac{3}{x-1}$ consequently becomes larger and larger. To put it accurately, however large a number be given us, we can take x so near 1 that the fraction $\frac{3}{x-1}$ will be larger than the given number. Similarly, we can say nothing whatever about the value of $\frac{x^2-1}{x-1}$ when $x = 1$, but we can say that as x approaches 1 the fraction approaches 2. When we say (figuratively) that the tangent of 90° is ∞, we mean first of all that this angle has no tangent. We also mean that we can take an angle so close to 90° that its tangent will be bigger than any number given us in advance.

In certain parts of the analytic geometry it would be greatly to the advantage of the student if he knew enough about quadratic inequalities to determine for what values of x a numerical

quadratic expression (such as $2x^2 - x - 3$) is positive and for what values it is negative.

In trigonometry a very troublesome deficiency on the part of many students is the lack of a *general* definition of the trigonometric functions. They do not know exactly what $\sin x$ means except for values of x between 0° and 90°. They should master a general definition that would hold for all angles, large or small, positive or negative.

The radian as a unit for measuring angles is not understood, as is shown by students asking why in some cases π stands for the number 3.1416 while in other cases it means 180°. Of course, π never means 180°, but means 3.1416 radians. The most important thing in this connection is the fact that when an angle is measured in radians, the length of a circular arc is equal to the angle subtended at the center multiplied by the radius. It is because of this fact that the radian replaces the degree entirely in the study of calculus.

A simple problem of frequent occurrence is that of finding the value of one function of an angle (except for the sign) when another function is given. Plenty of exercises, literal as well as numerical, should be given covering this point.

If possible, more time should be given to the solution of trigonometric equations. This is one of those subjects, theoretically simple but practically difficult, in which abundant practice is needed. The testing of roots in the original equation is especially important here, because the introduction of extraneous roots occurs so frequently.

And, finally, most of our students have only the vaguest notions about inverse trigonometric functions. These functions play an important part in the integral calculus. When one of two men working on a problem gets the result

$$\tan^{-1}\frac{5\tan\frac{x}{2} - 3}{4} - \tan^{-1} \tfrac{1}{2}$$

and the other gets

$$\tan^{-1}\left\{2\tan\left(\frac{x}{2} - \frac{\pi}{4}\right)\right\},$$

it does not even occur to them that their results may be the

same, but this is the case. To show that the one expression is equal to the other one is a more difficult problem of this kind than the students usually have to handle; but it is much easier than would appear at first sight, and they should know how to attack such a problem.

There are many students in engineering work who would have succeeded better in some other course; and the teacher of school mathematics has an important duty in this connection. He should strongly advise students who show no special aptitude for mathematics, and who have more ability along other lines, not to take engineering courses.

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In these near intimacies, we are ninety-nine times disappointed in our beggarly selves for once that we are disappointed in our friend; that it is we who seem most frequently undeserving of the love that unites us and that it is by our friends' conduct that we are continually rebuked and yet strengthened for a fresh endeavor.

We are all travellers in what John Bunyan calls the wilderness of this world—all, too, travellers with a donkey; and the best that we find in our travels is an honest friend. He is a fortunate voyager who finds many. We travel indeed to find them. They are the end and reward of life.—*Robert Louis Stevenson.*